# Numerical Simulation of Wooden Structures with Polymorphic Uncertainty in Material Properties

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**Abstract:** Due to nature, uncertainties are inherently present in structural parameters such as loadings, boundary conditions or resistance of structural materials. Especially material properties and parameters of wood are strongly varying in consequence of growth and environmental conditions. The considered uncertainties can be classified into aleatoric and epistemic uncertainty. To include this variation in structural analysis, available data need to be modelled appropriately, e.g. by means of probability and furthermore fuzzy probability based random variables or fuzzy sets. Therefore, a limited empirical data basis for Norway spruce, obtained by experiments according to DIN EN 408, is stochastically analysed including correlation, sensitivity analyses and statistical tests. In order to comprehend uncertainties induced by estimating the distribution parameters, the stochastic approach has been extended with fuzzy distribution parameters to fuzzy probability based random variables according to (Möller and Beer, 2004; Möller et al., 2000). To cope with epistemic uncertainties for e.g. geometric parameters of knotholes, fuzzy sets are used. The consequence for wooden structures is determined by fuzzy stochastic analysis (Götz et al., 2015) in combination with a FEM simulation using a model suitable for characteristics of a timber structure by (Jenkel and Kaliske, 2014). The uncertain results (e.g. displacements, failure loads) constituted by the proposed holistic approach – defining the material properties based on an empirical data basis and the attempt of representing the uncertainties in material parameters and methods itself – will be discussed in terms of further processing in engineering tasks.

**Keywords:** polymorphic uncertainty, fuzzy randomness, stochastic modelling, wood mechanics, structural analysis

## 1. Introduction

Wooden structures underlie a fundamental data uncertainty in every engineering related matter. The material parameters of wood are strongly varying due to the natural growth and environmental conditions even within small pieces of wood. These aspects hold especially true for the material parameters defined at a macroscopic level like the elasticity moduli and material strengths considered in this contribution. The reason for the variation at the macroscopic level might be found in the anatomical structure of wood including the cellular level. To incorporate these characteristics, wood can be described on the mesoscale including the growth layer dependent spatial variability of material parameters, see e.g. (De Amicis et al., 2011), or even on a micro and nanoscale, see (De Borst et al., 2013). However, the material parameters on the nano and microscale are naturally varying themselves. Regarding the design of timber structures according to EN 1995 (2010), global engineering material parameters are applied including further structural uncertainties e.g. in fibre orientation and knot hole size as well as distribution. Despite those uncertainties, it becomes necessary to assess the capabilities of a structure as well as characteristic variables for construction and design purposes. Methods for consideration of uncertainties in design of timber structures by means of randomness have been presented amongst others in (Fink and Köhler, 2014; Jenkel et al., 2015; Köhler et al., 2007; Spaethe, 1992).

In general, uncertainty can be classified into aleatoric and epistemic uncertainty. The combination of both yields polymorphic uncertainty, see e.g. (Götz et al., 2015). The first type includes e.g. the variations of material properties based on repetitive material tests. Due to a given amount of test results, the uncertainties can be represented by means of randomness, which in general satisfies statistical laws and possess a quasi objective information content. In this contribution, experiments performed on small specimens made of Norway spruce according to European standards are used as data basis. Epistemic uncertainties encounter non-statistical properties, information deficits and subjective influences. According to (Möller and Beer, 2004), epistemic uncertainties are further divisible into informal and lexical uncertainties. In this approach, geometric dimensions, knothole sizes and positions are defined as fuzzy sets according to the fuzzy set theory of (Zadeh, 1965).

Especially for insufficiently large observations, a statistical evaluation free of doubt is hard to constitute. Therefore, the use of fuzzy probability based random variables (*fp*-*r*), see (Götz et al., 2015; Pannier, 2011; Pannier et al., 2013), is proposed in order to encounter the uncertainty within the determination of stochastic parameters as well as representing the range of the response for deterministic fundamental solution. To approximate solutions of FE simulations, and analytical functions, artificial neural networks are hereby used as deterministic fundamental solution. To evaluate polymorphic uncertainty with respect to a wooden structure, a fuzzy stochastic analysis according to (Götz et al., 2015; Möller et al., 2007) is a valid approach, which yields to more realistic but uncertain result quantities.

This contribution is divided into five main sections. Firstly, a brief overview of mathematical fundamentals is given, regarding randomness, fuzziness and fuzzy randomness. The computational analysis procedure is explained as well. Approaches to model fuzzy distribution parameters for fuzzy probability based random variables are introduced. Hereafter, the data basis gathered from multiple experiments is presented together with the appropriate *fp*-*r* variables. Subsequently, a fuzzy stochastic analysis is applied on a wooden structure including knotholes.

## 2. Introduction of Uncertainty Models

The utilized uncertain structural analysis includes both, stochastic and fuzzy analysis. Therefore, the mathematical basis for each concept of uncertainty is introduced including the description of polymorphic uncertainty by means of fuzzy randomness.

### 2.1. Uncertainty models

In order to derive a numerical model for an adequate consideration of uncertainty, it is proposed to extend the common approach of stochastic modelling, used e.g. for material parameters, to fuzzy probability based random variables. Therefore, the fundamentals of randomness, fuzziness, covering epistemic uncertainty of e.g. geometric dimensions, and fuzzy probability based random variables are hereafter introduced, to incorporate the source and nature of present uncertainties.

#### 2.1.1. *Randomness*

A random variable $X$ is defined by the mapping $X:\ \Sigma \rightarrow \mathbb{R}$ fulfilling the condition

$$\forall I \in \mathcal{B}(\mathbb{R}) : X^{-1}(I) := \{\omega \in \Omega \mid X(\omega) \in I\} \in \Sigma \ , \tag{1}$$

whereas $\Omega$ correspondents with the set of elementary events $\omega$, $\Sigma$ is a $\sigma$-Algebra and $P$ is the probability measure, satisfying the probability axioms of Kolmogorov. The observation space is represented by $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ including the Borel-$\sigma$-Algebra $\mathcal{B}(\mathbb{R})$. If $X$ complies to the condition Eq. (1), an associated probability measure $P_X$ is furthermore defined as

$$P_X : \mathcal{B}(\mathbb{R}) \rightarrow [0,1] : I \mapsto P_X(I) = P(X^{-1}(I)) \ . \tag{2}$$

The underlying distribution of the random variable $X$ can be expressed by the cumulative distribution function $F_X$ (cdf) and its derivative called probability density function $f_X$ (pdf) for which holds

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t \ . \tag{3}$$

Based on the assumption that $f_X$ is continuous, the probability of an interval $I = [x_l, x_r]$ is related to $F_X$ as follows

$$P_X(I) = F_X(x_r) - F_X(x_l) \ . \tag{4}$$

The probability distribution of random variables is usually described by means of parametrized distributions $F_X(x, \theta)$. Common distribution types as the Normal or Log.-Normal distribution will be further represented as parametric model with respect to suitable distribution parameters $\theta$.

#### 2.1.2. *Fuzziness*

Considering a precise set $A \subseteq \mathbb{R}$, the characterizing function $\xi(\cdot)$, in terms of precise sets also called indicator functions (Viertl, 1996), is defined by

$$\xi_A : \mathbb{R} \rightarrow \{0;1\}, x \mapsto: \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \ . \tag{5}$$

Due to the imprecision of measurements, it becomes obvious that the definition of interval boundaries in this precise manner is a simple approach and not very realistic. Subsequently, the definition of precise sets has been enhanced to non-precise sets. For reasons of distinction, the characterizing function $\xi(\cdot)$ for non-precise data will be further expressed as membership function $\mu(\cdot)$ allowing an assessment of the membership to an domain between $[0,1]$. A fuzzy set $\tilde{A}$ can be expressed as set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, \mu_{\tilde{A}}(x) \geq 0\} \; . \tag{6}$$

To highlight the transformation from precise sets $A$ to uncertain sets in $\mathbb{R}$, fuzzy sets will be further referred to as $\tilde{A}$. The fuzzy set $\tilde{A}$ can also be referred to as a fuzzy number $\tilde{x}$, see (Möller and Beer, 2004). The membership function $\mu(\cdot)$ of an uncertain variable is a real function of a real variable with the following properties

$$\mu : \mathbb{R} \to [0,1] \, , \tag{7}$$

$$\exists \, x_0 \in \mathbb{R} : \mu(x_0) = 1 \, , \tag{8}$$

$$A_\alpha := \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq \alpha\} = [a_\alpha, b_\alpha] \, , \tag{9}$$

where the finite closed interval $A_\alpha$ is called $\alpha$-cut of $\mu(\cdot)$ (Zadeh, 1971). The $\alpha$-cut $A_0$ is called support of $\tilde{A}$. In this contribution, only convex fuzzy numbers according to (Möller and Beer, 2004; Pannier, 2011; Pannier et al., 2013) are considered. With respect to Eq. (9), all utilized fuzzy numbers are represented as either fuzzy triangular number

$$\tilde{x} = \langle a_0, x_0, b_0 \rangle \; , \tag{10}$$

or fuzzy trapezoidal interval number

$$\tilde{x} = \langle a_0, a_1, b_1, b_0 \rangle \; . \tag{11}$$

Both types of fuzzy numbers are illustrated in Fig. 1.

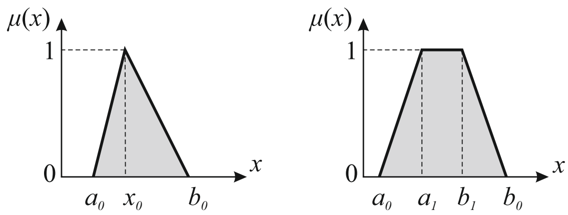



*Figure 1.* Fuzzy number $\tilde{x}$ as triangular and trapezoidal interval number.

2.1.3. *Fuzzy randomness*

The definition of fuzzy probability based random variables (*fp-r*) is founded on the assumption that the probability distribution of a random variable $X$ according to Eq. (2) cannot be described exactly due to a lack of information, see e.g. (Götz et al., 2015; Pannier, 2011; Pannier et al., 2013). Thus, a fuzzy probability distribution and a fuzzy probability space $(\Omega, \Sigma, \hat{P})$ can be introduced. The fuzzy probability $\hat{P}$ is represented as family of $\alpha$-cuts

$$\hat{P} = (P_\alpha)_{\alpha \in (0;1]} \; . \tag{12}$$

Each event $A \in \Sigma$ is related by $P_\alpha$ to an interval $[P_{\alpha,l}(A); P_{\alpha,r}(A)]$ for all $\alpha \in (0,1]$ such that the following condition is fulfilled

$$0 \leq P_{\alpha,l}(A) \leq P_{\alpha,r}(A) \leq 1 \ . \tag{13}$$

A fuzzy probability based random variable $X$ is defined by the mapping of the fuzzy probability space onto the observation space $X : \Omega \to \mathbb{R}$. The fuzzy probability distribution $\hat{P}_X$ is formulated as family of mappings $\hat{P}_X = ((P_X)_\alpha)_{\alpha \in (0;1]}$, with

$$(P_X)_\alpha : \mathcal{B}(\mathbb{R}) \to \{[l,r] \mid 0 \leq l \leq r \leq 1\} : \tag{14}$$

$$I \mapsto P_\alpha(X^{-1}(I)) = [P_{\alpha,l}(X^{-1}(I)), P_{\alpha,r}(X^{-1}(I))] \ . \tag{15}$$

The fuzzy probability distribution might be represented by a fuzzy cumulative distribution function $\hat{F}_X$, which is again defined as family of $\alpha$-cuts

$$\hat{F}_X = ((F_X)_\alpha)_{\alpha \in (0,1]} \tag{16}$$

$$(F_X)_\alpha = \{G : \mathbb{R} \to [0,1] \text{ cdf} \mid \forall x \in \mathbb{R} : \tag{17}$$

$$P_{\alpha,l}\left(X^{-1}((-\infty, x])\right) \leq G(x) \leq P_{\alpha,l}\left(X^{-1}((-\infty, x])\right)\} \ , \tag{18}$$

with an arbitrary cumulative distribution function $G(x)$. Each $G \in F_X$ is an original of $\hat{F}_X$. The applied cumulative distribution function $G$ is usually defined by distribution parameters $\theta$ in terms of $G(x, \theta)$. Then, the fuzzy cumulative distribution function $\hat{F}_X$ can be described with fuzzy distribution parameters $\tilde{\theta} = (\theta_\alpha)_{\alpha \in (0,1]}$. For example, a two parametric distribution function with parameters $\theta_1$ and $\theta_2$ yields

$$\hat{F}_X = (\{F_{\theta_1 \times \theta_2} \mid \theta_1 \in \tilde{\theta}_{1,\alpha}, \theta_2 \in \tilde{\theta}_{2,\alpha}\})_{\alpha \in (0,1]} \ . \tag{19}$$

This formulation is referred to as bunch parameter representation, since the fuzzy cumulative distribution and the fuzzy probability density function can be considered as assessed bunches of functions which are described by bunch parameters $\tilde{\theta}$.

## 2.2. Uncertain structural analysis

According to (Möller and Beer, 2004; Pannier, 2011), the so-called fuzzy stochastic analysis type I is utilized, in which the bunch parameter representation of fuzzy random variables is used. The general workflow is constituted in a three-loop computational model.

Initially, each input parameter $X_i$ with $i = 1, \ldots, n$ is represented by a fuzzy distribution according to Eq. (19), where each distribution type is determined individually for each input dimension, see Section 3. The fuzzy analysis, performed in the outer loop, implies the $\alpha$-discretization. A crisp space

$$\theta_{i,\alpha} = \{\theta_i \in \tilde{\theta}_i \mid \mu_{\tilde{\theta}_i}(\theta_i) \geq \alpha\} \subset \mathbb{R}^2 \tag{20}$$

is obtained for each $\alpha$-level, carried out on $\tilde{\theta}_i = (\tilde{\theta}_{1,i} \times \tilde{\theta}_{2,i})$. Evidently, every set of bunch parameters $\theta_i$ is associated with a trajectory $F_{\theta_i}(x)$, see Fig. 2. In the inner loop of the computational model, for each trajectory a stochastic analysis is performed, which concludes, in combination with a given deterministic fundamental solution $f_{\mathcal{Z}} : \mathbb{R}^n \to \mathbb{R}^m$, to an empirical distribution $\bar{F}_j(z)$ for each

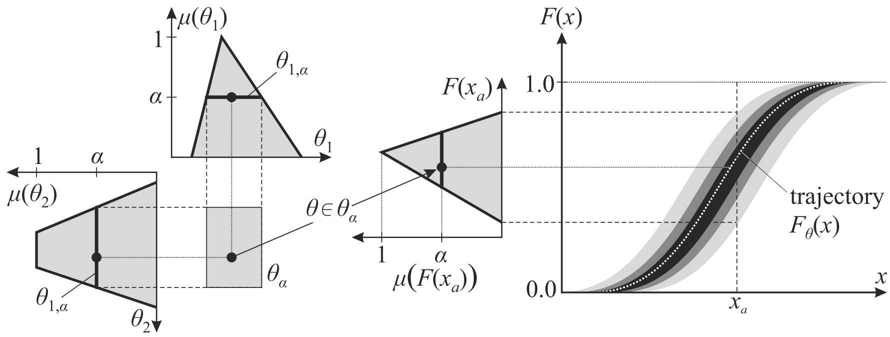



*Figure 2.* Trajectory regarding $\theta_\alpha$ (Möller and Beer, 2004).

result set $Z_j$ with $j = 1, \ldots, m$. To map the results at the appropriate $\alpha$-level, it becomes necessary to condense the empirical distribution $\bar{F}(z)$ to a representative scalar value $\sigma$. As information reduction method, any descriptive statistical evaluation parameters such as standard deviation, mean, median, quantiles etc. are conceivable for this task. In order to determine the bounds of the associated $\alpha$-level cut, a so-called $\alpha$-level optimization (Möller et al., 2000) is necessary to be carried for the computation of $\{\sigma_{\min,\alpha}, \sigma_{\max,\alpha}\}$ as illustrated in Fig. 3. Conclusively, the fuzzy result variable $\tilde{\sigma}$ is used to represent the uncertain results of the chosen information reduction method. Therefore, the choice of method should be related to the problem under investigation and considered in the final interpretation of the gained results.

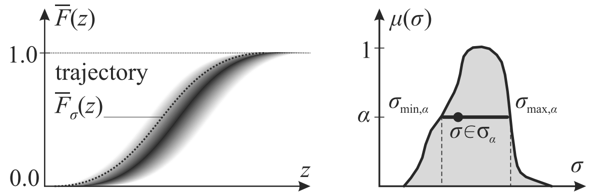



*Figure 3.* Distribution of $z$, representation with two reduction measurements (Möller and Beer, 2004).

## 3. Data Modelling

Since the proposed procedure of uncertain analysis is founded on *fp-r* variables for material parameters of wood, elements of statistics are necessary to determine stochastic parameters. Additionally, two approaches for modelling fuzzy distribution parameters as either triangular or trapezoidal interval numbers are introduced, whereas both methods base on the statistical evaluation of a given dataset.

### 3.1. Elements of mathematical statistics

To transform given test results into a distribution function, methods of inductive statistics are necessary. There are four distribution types considered in this contribution all of which are two-parametric such as $F_X(\theta_1, \theta_2)$. Beside the common Normal and Log.-Normal distribution, two extreme value distributions, Gumbel and Weibull, are considered. Point estimators are used to determine the distribution parameters. Methods of Moments (MoM) as well as Maximum-Likelihood Estimation (MLE) apply for the most common point estimation procedures (Köhler et al., 2007). The MoM is used since initial investigations revealed that especially for a small sample size of data the more robust MoM is sufficiently accurate considering the general variance within the data set itself.

The determination of a probability distribution function based on a given test results will be shown for the elasticity modulus in tangential direction $E_t$ and is initially based on 45 test samples $\underline{E}_t$. An examination with the boxplot detects potential outliers ($1.5\cdot$*inter quartile range*), see (Frigge et al., 1989). In order to determine the distribution parameters, the empirical mean value $\bar{m}(\underline{E}_t)$ and empirical standard deviation $\bar{\sigma}(\underline{E}_t)$ will be used.

For a real-valued continuous functions $f(x)$, the $n$-th moment is determined according to (Spaethe, 1992)

$$\mu_n = \int_{-\infty}^{\infty} (x-c)^n f(x) dx \; . \tag{21}$$

The mean value $\mu$ of $f(x)$ is equal to the first moment (with $c = 0$) and the variance $\sigma^2$ satisfies the second moment if $c = \mu$ (central moment). The moments depending on the parameters $\theta_1, \theta_2$ can be expressed as

$$\begin{aligned} \mu_1 &= g_1(\theta_1, \theta_2) \\ \mu_2 &= g_2(\theta_1, \theta_2) \; . \end{aligned} \tag{22}$$

The MoM defines the empirical moments $\bar{m}, \bar{\sigma}^2$ equal to the moments $\mu_1, \mu_2$ of true distribution with the result that the equation system

$$\begin{aligned} \bar{m} &= g_1(\hat{\theta}_1, \hat{\theta}_2) \\ \bar{\sigma}^2 &= g_2(\hat{\theta}_1, \hat{\theta}_2) \end{aligned} \tag{23}$$

leads to the parameter estimations $\hat{\theta}_1, \hat{\theta}_2$ for $\theta_1, \theta_2$. For continuous functions, the estimators based on the MoM fulfil the following properties, such as consistency

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \varepsilon) = 0 \;, \forall\, \varepsilon > 0 \;, \tag{24}$$

as well as being asymptotically unbiased

$$\lim_{n\to\infty} \mathrm{E}(\hat{\theta}) = \theta \; . \tag{25}$$

To test the conformity of the empirical distribution with an assumed distribution type, two statistical tests are performed, see e.g. (Rinne, 2008; Viertl, 1997), the Kolmogorov-Smirnov test (KS) and the $\chi^2$-test with focus on the KS test due to the sample size of $n^* = 44$. Both statistic tests belong to the category "Goodness of fit", which determines how well an empirical distribution suits a hypothetical distribution function. The KS test is based on the maximum difference between an empirical and hypothetical cumulative distribution, with the given KOLMOGOROV-SMIRNOV statistic for the null-hypothesis $H_0 : F_n(x) = F_0(x)$ for a continuous probability distribution

$$D_n = \sup_x |F_n(x) - F_0(x)| \ . \tag{26}$$

The critical value of the maximum difference with respected to the significance level $\alpha$ is approximately (for $n \geq 35$)

$$c_\alpha = \frac{\sqrt{\ln(\frac{2}{\alpha})}}{\sqrt{2n^*}} \ , \tag{27}$$

hence the null hypothesis will be rejected if $D_n > c_\alpha$ (Messay, 1951). Therefore, $F_0(x)$ can be assumed as the underlying distribution function for the empirical distribution with a certain sureness based on $\alpha$, see Fig 4(b). The chosen criteria for rejecting the null hypothesis with the $\chi^2$-test is the $p$-value $\chi^2_p$ of the $\chi^2$-test statistic (Sellke et al., 2001). If $\chi^2_p < \alpha$, the hypothetical distribution is rejected.

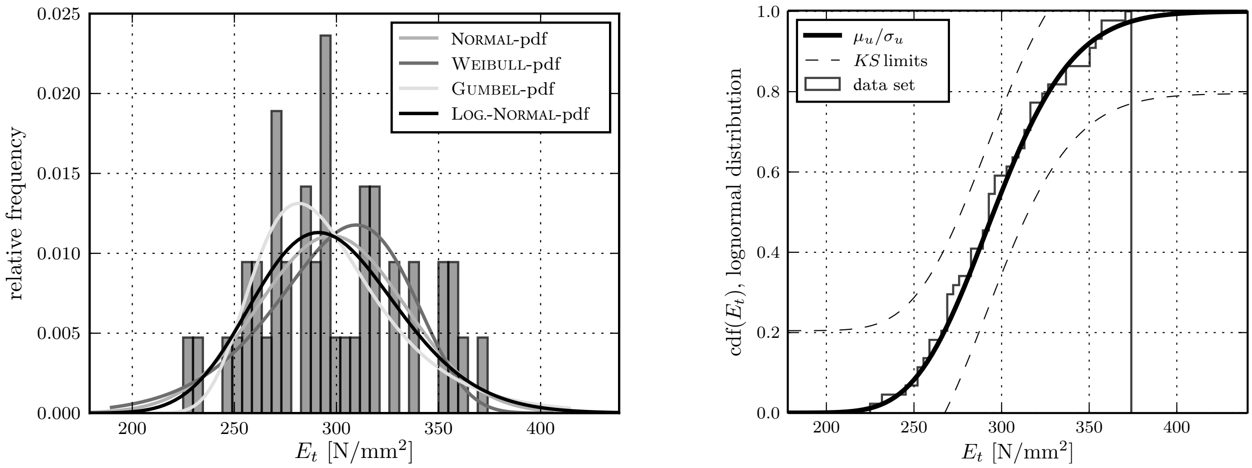
Table I. Distribution parameters for $E_t$.

| | GUMBEL | LOG.-NORMAL | NORMAL | WEIBULL |
|---|---|---|---|---|
| KS ($c_{0.05}$) | 0.2047 | 0.2047 | 0.2047 | 0.2047 |
| KS ($D_n$) | 0.0862 | 0.0670 | 0.1412 | 0.1357 |
| $\chi^2$ ($p$-value) | 0.8364 | 0.8367 | 0.5893 | 0,5036 |
| parameter 1 | $a = 0,0356$ | $\mu_u = 5,6886$ | $\mu = 297,63$ | $\theta = 312,90$ |
| parameter 2 | $b = 281,45$ | $\sigma_u = 0,1203$ | $\sigma = 35,937$ | $k = 9,9617$ |

As Table I shows, neither of the four distribution types is declined by both statistical tests, which yields to the assumption that all of the distributions are valid approximations for the test samples. Since the maximum absolute difference between the LOG.-NORMAL distribution and the empirical distribution is the lowest, a LOG.-NORMAL distribution ($\sigma_u, \mu_u$, see Tab. I) can be considered as best fitting for $\underline{E_t}$. The four distribution types, represented by the probability density functions (parameters see Tab. I) are shown in Fig. 4(a) with the relative frequencies of $x \in \underline{E_t}$. The small positive skewness of the histogram might lead to the preference of the GUMBEL and LOG.-NORMAL distributions by the statistical tests.

### 3.2. Principles of modelling the fuzzy probability based random variables

Since multiple admissible distributions exist to describe an empirical distribution, it is likely that the chosen point estimators are not the most accurate way to determine the distribution parameters




(a) Probability density functions and histogram of $\underline{E}_t$

(b) LOG.-NORMAL cumulative distribution function for $E_t$ and empirical cumulative distribution of $\underline{E}_t$, limits for KS statistic Eq. (27)

*Figure 4.* All admissible distributions for $E_t$ and the best fitting distribution within KS limits for $\alpha = 0.05$.

with respect to the uncertainty in the data. To define the bunch parameters of any chosen two-parametric fuzzy random based random variable

$$F_X = (\{F_{\theta_1 \times \theta_2} \mid \theta_1 \in \tilde{\theta}_{1,\alpha}, \theta_2 \in \tilde{\theta}_{2,\alpha}\})_{\alpha \in (0,1]} \ , \tag{28}$$

for $\alpha = 0$ and $\alpha = 1$, one of the following methods is used. A common approach for $\alpha = 0$ is based on interval estimation of a parameter $\theta$. Assuming $\theta$ can vary within a defined confidence interval $[\vartheta_{\min}, \vartheta_{\max}]$

$$P_\theta(\vartheta_{\min} < \theta < \vartheta_{\max}) = (1 - \alpha) \ , \tag{29}$$

a set of the minimum and maximum values of parameters, according to the confidence level $(1 - \alpha_s)$ might be used describing a set of distribution functions, see e.g. (Viertl, 1997). According to Eq. (28) and (29), the bounds of the support are defined as follows

$$\theta_{0,l} = \vartheta_{\min} \tag{30}$$

$$\theta_{0,r} = \vartheta_{\max} \, . \tag{31}$$

As described, one fuzzy parameter must hold $\{\exists\ \vartheta_0 \in \tilde{\theta} \mid \mu_{\tilde{\theta}}(\vartheta_0) = 1\}$. To obtain a so-called fuzzy triangular number, point estimators such as MLE, MoM are appropriate to gain $\vartheta_0$. Therefore, the uncertain parameter will be further expressed as

$$\tilde{\theta}_{(\cdot)} = \langle \vartheta_{\min}, \vartheta_0, \vartheta_{\max} \rangle_{(\cdot)} \ . \tag{32}$$

Considering any point estimation of stochastic parameters for an empirical distribution with low sample sizes, it becomes obvious that this definition is accompanied by uncertainty as well. Consequently, an alternative definition to the one previously propagated must be found for input dimensions with higher sensitivity relative to the investigated output dimension. Therefore, the error made by the point estimation has to be taken into account. Under the assumption that an admissible distribution type $F_X(\theta_1, \theta_2)$ fits the data set $X$, a sufficient amount of $n > 10^6$ samples

$$\bar{x} \in \bar{X}, P(\bar{X}) = F_X(\theta_1, \theta_2) \ , \tag{33}$$

should substitute the entire data set. To determine the deviation of the admissible parameter set from the actual provided data, the variation of mean and standard deviation of an extended original data set is computed with

$$X_i^* = X \cup \{\bar{x}_i\} \ \forall \ \bar{x}_i \in \bar{X} \ , \ i = \{1, \ldots, n\} \ , \tag{34}$$

$$\mu_i = \mathrm{E}[X_i^*] \ , \tag{35}$$

$$\sigma_i = \sqrt{\mathrm{Var}[X_i^*]} \ . \tag{36}$$

Hence, $\boldsymbol{\mu}^* = [\mu_1, \ldots, \mu_n]$ as well as $\boldsymbol{\sigma}^* = [\sigma_1, \ldots, \sigma_n]$ are two data sets containing the deflections of the first and second statistical moments. Concerning the longitudinal stiffness $E_l$, whereas the Weibull distribution can be considered as admissible, the distribution of the extended mean values (see Eq. (35)) is shown in Fig. 5(a). To evaluate the imprecision, the empirical quantiles $\mu_{q_5}$ and $\mu_{q_{95}}$ are chosen. With the same empirical quantiles of $\boldsymbol{\sigma}^*$ and the MoM, two parameter sets can be obtained

$$\text{MoM} : \begin{cases} (\mu_{q_5}, \sigma_{q_5}) & \rightarrow (\vartheta_{1,\min_1}, \vartheta_{1,\min_2}) \\ (\mu_{q_{95}}, \sigma_{q_{95}}) & \rightarrow (\vartheta_{1,\max_1}, \vartheta_{1,\max_2}) \end{cases} \ . \tag{37}$$

Consequentially for each distribution parameter, a fuzzy trapezoidal interval number such as

$$\tilde{\theta}_{(\cdot)} = \langle \vartheta_{\min}, \vartheta_{1,\min}, \vartheta_{1,\max}, \vartheta_{\max} \rangle_{(\cdot)} \tag{38}$$

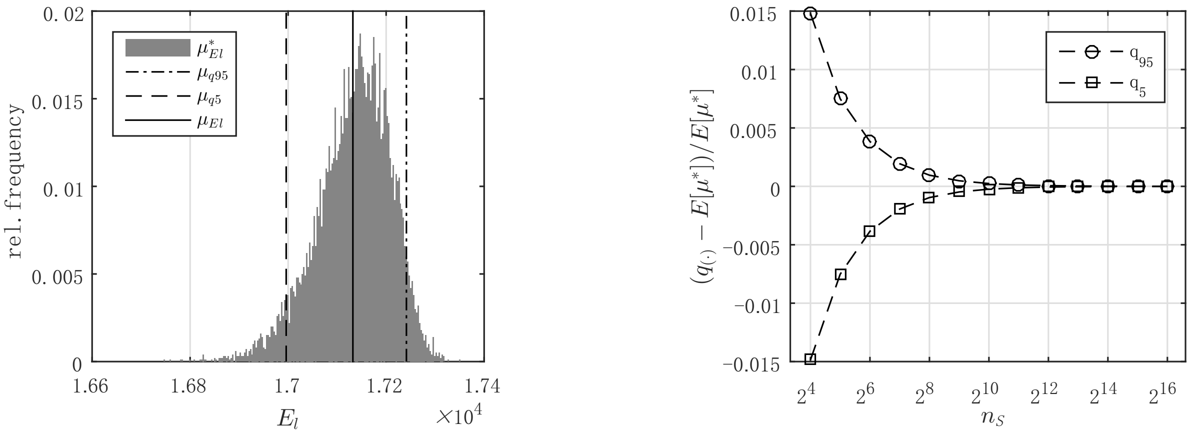
can be defined, whereas each $\alpha$-cut $A_1$ is bounded by the corresponding distribution parameters in Eq. (37). The procedure is applied on material parameters of wood in Section 4. To observe the convergence properties, the fitting of admissible distribution parameters is performed on increasing samples size $n_S$ for an arbitrary data set. In Fig. 5(b), a normalized error

$$\varkappa_{(\cdot)} = \frac{(q_{(\cdot)} - E[\mu^*])}{E[\mu^*]} \tag{39}$$

in relation to $n_S$ is shown. Hence, based on the condition that for large $n_S$ an admissible function exists, the following applies

$$\lim_{n_S \rightarrow \infty^+} \varkappa_{(\cdot)} = 0 \ , \tag{40}$$

as Fig. 5(b) illustrates. In terms of fuzzy numbers for stochastic parameters it yields $\vartheta_{1,min} = \vartheta_{1,max} = \vartheta_0$, whereby the fuzzy trapezoidal interval is reduced to a fuzzy triangular number.



(a) Distribution of $\boldsymbol{\mu}^*$

(b) Convergence behaviour

*Figure 5.* Exemplary distribution of $\boldsymbol{\mu}^*$ (based on $\underline{E}_l$) and convergence behaviour of $\varkappa_{(\cdot)}$ related to increasing $n_S$.

## 4. Parameter Modelling and Data Basis

The introduced methods are applied to model the uncertainty of material parameters of wood subsequently. The mechanical behaviour of the anisotropic material wood differs significantly in the material directions radial $r$, tangential $t$ and longitudinal $l$ and strongly depends on the type of loading like tension $t$, compression $c$ and shear loading $v$. In the structural analyses presented below, macroscopic material parameters taking into account these dependencies are utilized. The material parameters and their uncertain distributions are modelled on the basis of empirical data.

### 4.1. Data base situation

The data base applied in this contribution has been obtained in experiments described in (Jenkel et al., 2015; Ulrich and Seim, 2014). The investigated material parameters include the elasticity moduli and the material strengths in the material directions and depending on the type of loading. The tests have been carried out on small specimen, as far as possible free of inhomogeneities, according to European and German standards given in Tab. II. In the table, the number of samples (under consideration of outliers) as well as the empirical mean value $\bar{m}$ and standard deviation $\bar{\sigma}$ are given for each parameter. The total data sets are documented in (Jenkel et al., 2015; Ulrich and Seim, 2014).

The experiments are designed to take all samples independently on the basis of identical conditions (*i.i.d.* paradigm). For the parameters given in Tab. II, data sets should be obtained disregarding the interaction to other parameters. Especially density and moisture, as most relevant parameters influencing all material parameters of wood, should be blinded out. Thus, as far as possible, specimen of comparable density are used. The moisture is conditioned in climate chambers. To avoid size effects and obtain comparable parameters, identical specimen measures are applied for most of the test series. Specimen with equal material directions are used in each test series considering all three material directions, except for the tensile strength perpendicular to grain.

Table II. Experimental basis according to (Jenkel et al., 2015; Ulrich and Seim, 2014).

| parameter | $E_r$ | $E_t$ | $E_l$ | $f_{t,90}$ | $f_{t,l}$ | $f_{c,r}$ | $f_{c,t}$ | $f_{c,l}$ | $f_v$ |
|---|---|---|---|---|---|---|---|---|---|
| standard | DIN 52192 | DIN 52192 | DIN 52185 | EN 408 | DIN 52188 | DIN 52192 | DIN 52192 | DIN 52185 | EN 408 |
| samples | 41 | 44 | 28 | 30 | 30 | 45 | 45 | 30 | 30 |
| $\bar{m}$ $[N/mm^2]$ | 656 | 298 | 17132 | 2.64 | 121.64 | 3.09 | 3.64 | 43.60 | 5.77 |
| $\bar{\sigma}$ $[N/mm^2]$ | 107 | 36 | 2211 | 0.33 | 18.20 | 0.23 | 0.43 | 2.07 | 0.73 |

### 4.2. Data modelling

The available data is evaluated statistically to model the material parameters as fuzzy probability based random variables. The best fitting distribution types for the data sets used in the examples presented below are given in Tab. III. Exemplarily, the longitudinal elasticity moduli $E_l$ is represented by means of a Weibull distribution

$$\widehat{F}_{E_l} = \left( \left\{ F_{\theta \times k} \,|\, \theta \in \widetilde{\theta}_\alpha,\ k \in \widetilde{k}_\alpha \right\} \right)_{\alpha \in (0;1]}, \tag{41}$$

whereas the distribution parameters are modelled as fuzzy trapezoidal interval numbers according to Tab. III.The distribution is illustrated in Fig. 6. The black graphs in Fig. 6(a) are obtained using the *max*- and *min*-sets of the space of bunch parameters defined at $\alpha = 1$ marked by I and II in Fig. 6(b). The light grey graphs are computed with $\alpha = 0$ for the *max*- and *min*-sets of the bunch parameters marked by III and IV.

## 5. Example

The methods presented above are used subsequently to compute the ultimate load of a timber board containing knots at tensile loading are analysed under consideration of uncertainties in material and geometrical parameters. The material parameters are modelled as fuzzy probability based random variables (*fp-r*) while geometric parameters are described by fuzzy variables.

Knots in timber are remnants of branches in trees and can be considered as structural inhomogeneities. The size of knots and the boundary to the surrounding wood can often not be identified exactly. Thus, the size of the knots is regarded as being uncertain and modelled by means of fuzzy numbers. The board with dimensions $t \times b \times l = 18 \times 150 \times 350\,mm$ is analysed at uniform tensile loading as shown in Fig. 7. The aim is to compute the ultimate load. An FE analysis according to (Jenkel and Kaliske, 2014) is applied to generate an artificial neural feed forward network, which is used as deterministic fundamental solution $f_{\mathcal{Z}}$. Thereby, the knots and the surrounding wood are

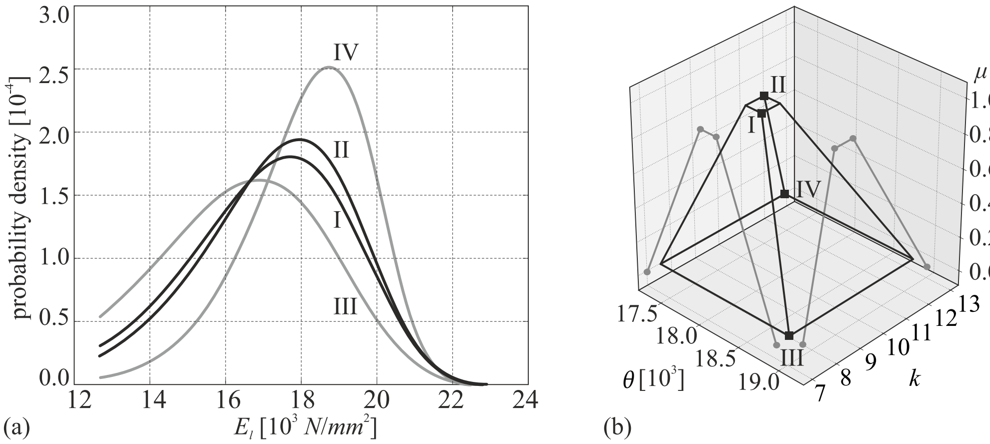



*Figure 6.* Fuzzy probability based random variable $E_l$: (a) fuzzy probability density function $\widehat{f}_{E_l}$ and (b) Cartesian product $\{\widetilde{\theta} \times \widetilde{k}\}$.

Table III. Evaluation of experimental data (all data in $[N/mm^2]$).

| parameter | distribution type | fuzzy distribution parameters |
|---|---|---|
| $E_l$ | Weibull | $\widetilde{\theta} = \langle 17256, 17790, 18326, 18785 \rangle$<br>$\widetilde{k} = \langle 7.529, 9.283, 10.99, 13.83 \rangle$ |
| $f_{t,90}$ | Weibull | $\widetilde{\theta} = \langle 2.6627, 2.7771, 2.8879 \rangle$<br>$\widetilde{k} = \langle 7.7578, 9.7439, 13.5851 \rangle$ |
| $f_{t,l}$ | Normal | $\widetilde{\mu} = \langle 114.45, 121.11, 125.37, 132.02 \rangle$<br>$\widetilde{\sigma} = \langle 14.491, 18.134, 21.248, 27.354 \rangle$ |
| $f_{c,r}$ | Gumbel | $\widetilde{a} = \langle 2.9097, 2.9687, 3.0375 \rangle$<br>$\widetilde{b} = \langle 4.9700, 5.9334, 7.7054 \rangle$ |
| $f_{c,l}$ | Log.-Normal | $\widetilde{\mu}_u = \langle 3.7554, 3.7737, 3.7917 \rangle$<br>$\widetilde{\sigma}_u = \langle 0.0395, 0.0491, 0.0559 \rangle$ |
| $f_v$ | Log.-Normal | $\widetilde{\mu}_u = \langle 1.6979, 1.7439, 1.7829 \rangle$<br>$\widetilde{\sigma}_u = \langle 0.0931, 0.1259, 0.1553 \rangle$ |

not distinguished by element edges, but by means of integration points. This smeared FE model is feasible since an individual coordinate system and material parameters can be assigned to every integration point of each finite element, compare (Zohdi and Wriggers, 2005). The board is simply discretized by a regular mesh with $2 \times 12 \times 30$ hexahedral 8-node finite elements, see Fig. 7. To improve the approximation of knots, three integration points are used in each direction per element. Due to the indirect representation of knots in the FE model, a fixed regular mesh can be applied

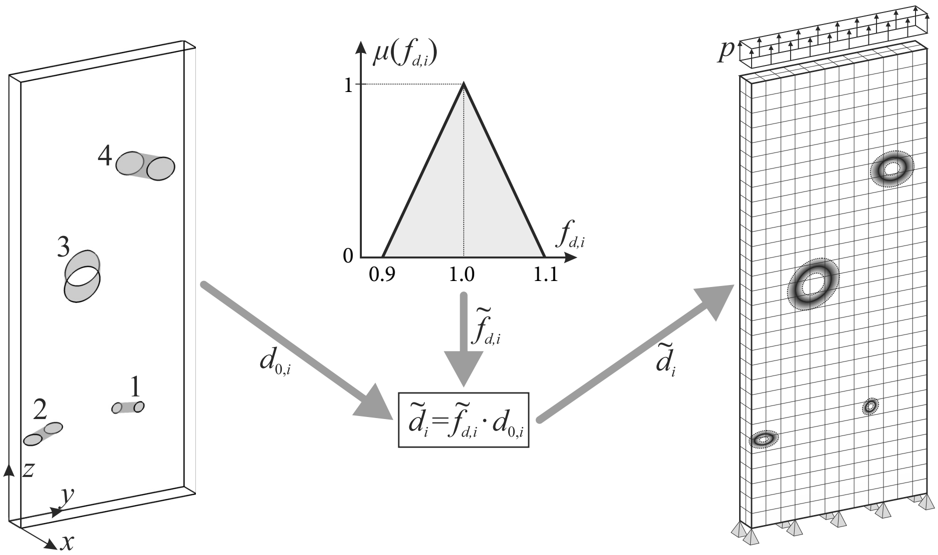



*Figure 7.* Original geometrical model and FE model with fuzzy sized knots due to fuzzy knot diameter $\tilde{d}_i$.

in the uncertain analysis. Otherwise, a new mesh would need to be generated for each solution step due to changing knot size.

The board analysed here is experimentally investigated in (Stübi, 2001), whereas the knots are documented in size and position on the board surfaces. The procedure how to derive a geometrical model as shown in Fig. 7 from these measurements is presented in (Jenkel and Kaliske, 2014). For the given example, the knots are described as cylinders passing the board in different angles. The board contains 4 knots $i = \{1, 2, 3, 4\}$. The original knot diameters $d_{0,i}$ taken from (Jenkel and Kaliske, 2014; Stübi, 2001) are varied in-between ±10%. Thus, four fuzzy knot diameters

$$\tilde{d}_i = \tilde{f}_{d,i} \cdot d_{0,i}, \quad i = \{1, 2, 3, 4\} \tag{42}$$

are introduced using fuzzy triangular numbers $\tilde{f}_{d,i} = \langle 0.9, 1.0, 1.1\rangle$ as knot factors, see Fig. 7.

Before the structural analysis is carried out, a material coordinate system representing the three material directions $r$, $t$ and $l$ needs to be assigned to every integration point. In general, the longitudinal direction is defined by the stem direction, the tangential direction by the growth rings and the radial direction by the medullary rays pointing to pith. In the area of branches, the fibre course, i.e. the longitudinal direction, is deviating from the stem direction. Therefore, the fibre course is computed by means of a streamline approach presented in (Jenkel and Kaliske, 2014) based on a flow-grain analogy. Since the knot diameters are varied, this computation has to be carried out for every solution step within the fuzzy stochastic structural analysis.

The ultimate load $p_u$ is computed using a Tsai-Wu plasticity formulation with linear isotropic softening, see e.g. (Schmidt and Kaliske, 2009; Tsai and Wu, 1971). In the simulations, the load $p$ is increased by small increments. If the total cross-section in an arbitrary region of the board is in the plastic regime, the load will decrease. The ultimate load is determined as maximum loading in the computed load-displacement dependencies $p_u = \max(p)$.

The material parameters are modelled based on the empirical data described above. In the applied elasto-plastic material model, the 9 material parameters described in Tab. II ($E_l$, $E_r$, $E_t$, $f_{c,r}$, $f_{c,t}$, $f_{c,l}$, $f_{t,90}$, $f_{t,l}$ and $f_v$) are utilized. In addition, the shear moduli, POISSON's ratios and the rolling shear strength are needed. Since these parameters are not investigated in the experiments, deterministic standard values are chosen in terms of $\nu_{rt} = 0.24$, $\nu_{tl} = \nu_{rl} = 0.45$ and $G_{rt} = 80\,N/mm^2$, $G_{tl} = G_{rl} = 800\,N/mm^2$ and $f_{v,rt} = 0.1 \cdot \bar{m}_{f_v}$ for all simulations.

Two configurations are considered, whereas the knots are modelled as holes ($k_{type} = 1$) and as being filled and fully connected to the surrounding wood ($k_{type} = 2$). If $k_{type} = 1$, the knot holes are approximated in the regular mesh using 1% of the values of the material parameters applied for the surrounding wood. If $k_{type} = 2$, the same material parameters as for the surrounding wood are applied for the integration points inside the knots, but with material coordinate systems defined by the longitudinal branch axes. The influence of the 9 material parameters and the four knot factors has been investigated on the basis of a Design of Experiments (DoE) for both configurations, $k_{type} = 1$ and $k_{type} = 2$. Beside the ultimate load $p_u$, the displacement $u_z(p_u)$ at the point of maximum load, obtained as mean value of the displacements in $z$-direction of all loaded nodes, is regarded as result quantity. The results are evaluated in sensitivity analyses using SOBOL indices (Sobol, 2001). In Fig. 8, the sensitivity measures of all input quantities are given.

The outcomes are similar for both knot configurations. The ultimate load $p_u$ is influenced mostly by $f_{t,l}$ and $f_v$. The displacement at the point of ultimate load $u_z(p_u)$ is affected by $E_l$, $f_{t,l}$, $f_v$ and slightly $f_{t,90}$. The influence of the knot factors seems to be rather small. The effects of the different factors might neutralize if one knot becomes larger while the other gets smaller. If $k_{type} = 2$, the knots have larger influence, probably due to their load bearing capacity. The SOBOL indices of the particular knot factors correspond to the original knot size. As a consequence of the sensitivity

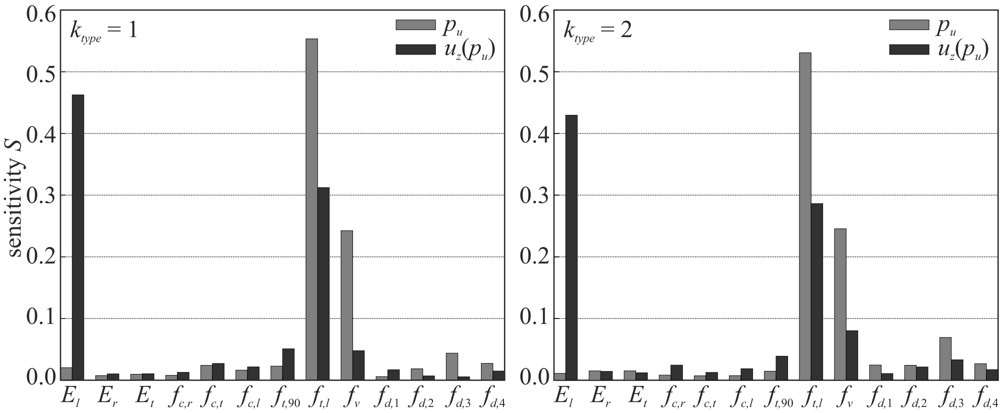



*Figure 8.* Sensitivity measures $S$ of all input variables regarding the result values $p_u$ and $u_z(p_u)$.

analysis, the longitudinal elasticity modulus $E_l$, the tensile strength perpendicular to grain $f_{t,90}$, the longitudinal tensile strength $f_{t,l}$ and the shear strength $f_v$ are modelled as fuzzy probability based random variables using the best fitting distribution types identified in Tab. III. All other material parameters are modelled deterministically by their mean values according to Tab. II. A WEIBULL distribution is used to represent $E_l$, see Fig. 6. For $f_{t,l}$, a NORMAL distribution fits best to the empirical data. Since these material parameters appear to be most relevant, the distribution parameters are described by means of fuzzy trapezoidal interval numbers, see Tab. III.

For each configuration ($k_{type} = 1, 2$), the ultimate load $p_u$ is computed as fuzzy stochastic result quantity. Quantiles $\bar{q}_i$ are chosen to represent the uncertain distribution of $p_u$ here. The $\alpha$-level optimization is carried out on $\alpha$-levels $\alpha = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$ regarding $\bar{q}_1$, $\bar{q}_5$, $\bar{q}_{25}$, $\bar{q}_{50}$, $\bar{q}_{75}$, $\bar{q}_{95}$ and $\bar{q}_{99}$. The results of the simulation with $k_{type} = 1$ are illustrated in Fig. 9. The uncertain ultimate load $p_u$ computed using $k_{type} = 2$ is depicted in Fig. 10.

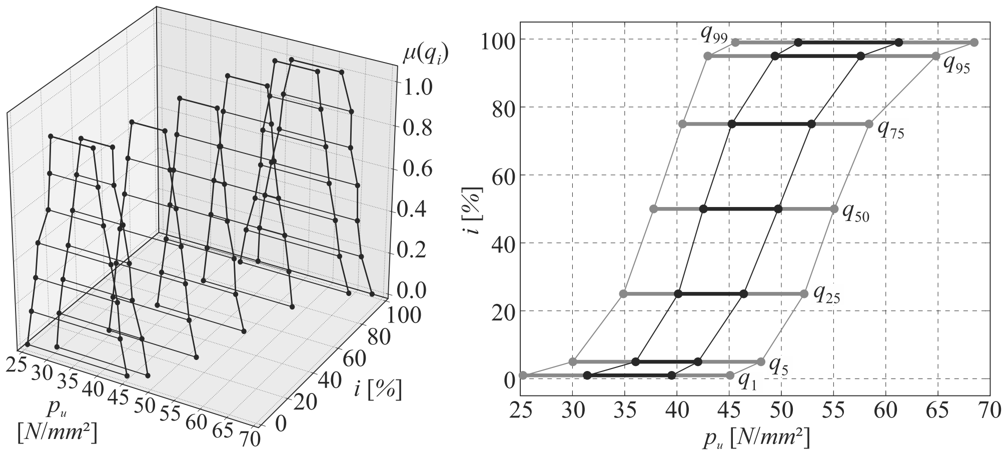



*Figure 9.* Uncertain distribution of $p_u$ represented by fuzzy quantiles $\tilde{\bar{q}}_i$ for $k_{type} = 1$.

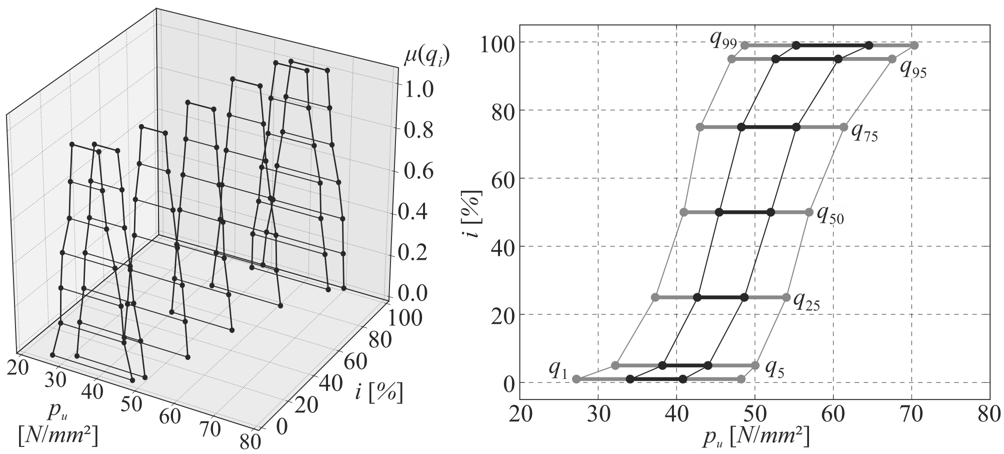



*Figure 10.* Uncertain distribution of $p_u$ represented by fuzzy quantiles $\tilde{\bar{q}}_i$ for $k_{type} = 2$.

In both figures, the fuzzy numbers for the particular quantiles $\tilde{\bar{q}}_i$ with $i = \{1, 5, 25, 50, 75, 95, 99\}$ are depicted in 3D views. The black circles mark the bounds of each fuzzy number on each $\alpha$-level. All fuzzy quantiles are obtained as kind of fuzzy trapezoidal interval numbers. In addition, a top view is given. The black lines represent the plateaus of the fuzzy trapezoidal interval numbers while the grey lines represent the support. As can be seen in the top views, the uncertain distribution function of $p_u$ is approximated by the quantiles.

For $k_{type} = 1$, smaller ultimate loads are computed as for $k_{type} = 2$, which is reasonable due to the load bearing capacity of the knots. The membership functions of the fuzzy quantiles are similarly shaped for both configurations. The range of the intervals computed for each quantile on each $\alpha$-level is comparable for $k_{type} = 1$ and $k_{type} = 2$. Moreover, the uncertain distributions according to the top views in Figs. 9 and 10 are very similar, except that the distribution for $k_{type} = 2$ seems to be shifted to the right about 2-3 $N/mm^2$.

In this example, methods for the consideration of material and structural inhomogeneities are applied jointly revealing the advantages of the introduced uncertainty models. The results given in Figs. 9 and 10 include information, which could not have been achieved by application of a pure stochastic analysis.

## 6. Conclusion and Outlook

The uncertainty of material and structural parameters has manifold reasons. If the uncertainty of input parameters shall be considered realistically in a structural analysis, appropriate data models are needed. In this contribution, a general description with polymorphic uncertainty is utilized and further classified into aleatoric and epistemic uncertainty. The first type is described using randomness while the latter is represented by means of fuzziness. The combination of both yields fuzzy randomness which is ideally suited to describe the uncertainty of material parameters of wood. Although the natural variability of material parameters is identified with aleatoric uncertainty, an application of randomness is often not feasible due to the limitations of available data bases.

Methods to model empirical data by fuzzy randomness with focus on fuzzy probability based random variables are introduced. The procedure is applied to describe the uncertainty of macroscopic material parameters of wood. In addition, the uncertainty of geometrical and further structural parameters is represented using fuzzy variables. The procedure of a fuzzy stochastic structural analysis is described theoretically and demonstrated by an examples. In the structural analyses of wooden structures, information is considered which could not have been included in stochastic analyses nor a single deterministic structural analysis.

According to EN 1990 (2010), a semi-probabilistic safety concept is proposed for the future determination of partial safety factors. A stochastic analysis procedure as special case of the introduced methods can be used for the calibration of safety factors in a partial factor design concept. A failure probability might be prescribed, which is used to determine deterministic design values from the uncertain input parameters and results. These values simply need to be related to the characteristic values to define partial safety factors.

Although the uncertainty models fuzziness and fuzzy randomness are established in science, an application for the purpose of standardization seems improbable in short term. Actually, engineers demand a further simplification of the design rules instead of a wider range of methods (Seim et al., 2012). However, the evaluation of the uncertain results can give an additional input for the determination of less conservative safety factors and a better utilization of the load bearing capacity of timber structures. Future work is necessary to get from the fuzzy stochastic structural analysis presented here to recommendations for a numerical design concept.

In uncertain structural analyses, uncertain results are computed containing all information provided due to the uncertain input parameters. Engineers require deterministic values to determine a structural design. Measures of central tendency as mean value and measures of dispersion as quantiles can be used to reduce the information of uncertain variables to deterministic values. Similar measures have been used in the fuzzy stochastic structural analysis to describe the uncertain distributions of the results. Due to the application of fuzzy randomness, the mean values and quantiles are obtained as uncertain quantities, which might need to be simplified themselves. Information reducing measures for fuzzy numbers are introduced e.g. in (Beer and Liebscher, 2008; Graf et al., 2009).

In this contribution, the idea is not to reduce the uncertainty but keep as much information as possible. If all available data are considered in the structural analyses, the influence of the uncertainty of the input parameters on the structural results can be evaluated. Engineers can obtain indication which input parameters deserve closer attention and might be modified to improve a structural design. Therefore, robustness analyses, such as in (Beer and Liebscher, 2008; Graf et al., 2009), and sensitivity analysis, see e.g. (Pannier and Graf, 2015), are powerful tools. Moreover, the procedures presented in this contribution can be used in structural optimization approaches as introduced in (Götz et al., 2015).

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